Contact manifolds with symplectomorphic symplectizations

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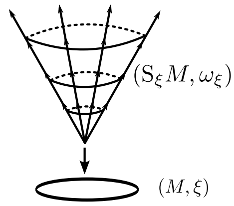
What is symplectization ?

Contact manifold (M,ξ) —Symplectization→ Symplectic manifold $(\mathrm{S}_\xi M, \omega_\xi)$

Examples: Standard contact sphere S^{2n-1} → Symplectic vector space minus zero $\mathbb{R}^{2n}\setminus\{0\}$

Sphere tangent bundle ST^*M → Cotangent bundle minus zero section $\mathrm{T}^*M\setminus 0_M$



Definition: $\mathrm{S}_\xi M = \{\beta\in \mathrm{T}^*M \mid \ker\beta = \xi\}$

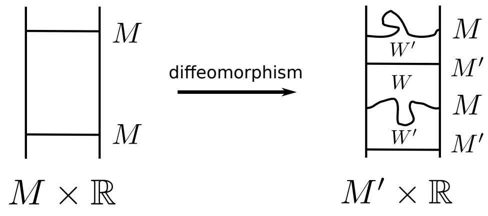
Properties:
- Symplectic submanifold of T^*M
- $\mathbb{R}$ -Principal bundle. Symplectic dilation: $\phi_t^*\omega_\xi = e^t\omega_\xi$
- Diffeomorphic to $\mathbb{R}\times M$

Conclusion: Contact geometry = $\mathbb{R}$-equivariant symplectic geometry of the symplectization

Question: If two contact manifolds have symplectomorphic symplectizations, are they contactomorphic ?

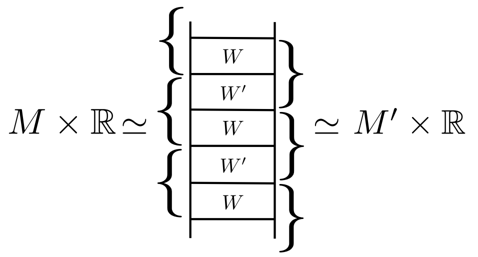
The Mazur trick

Suppose $M\times\mathbb{R}$ and $M'\times\mathbb{R}$ are diffeomorphic:



Get "inverse" cobordisms W and W'

Conversely, given such cobordisms, apply the Mazur trick:

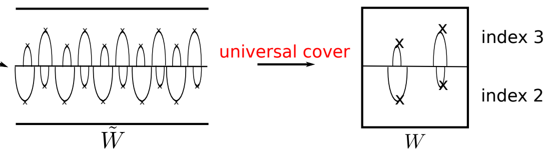


Such cobordisms are h-cobordisms. If $\dim M = m\geq 5$, Morse theory allows to classify them in terms of Whitehead torsion.

What is Whitehead torsion?

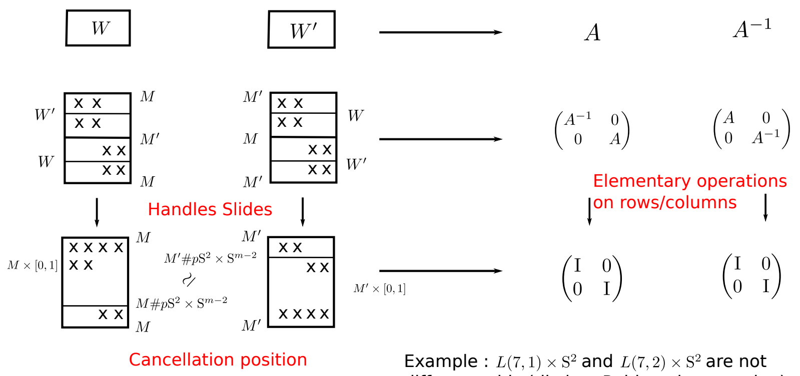
Normal form lemma : W admits a handle decomposition with index 2 and 3 handles.

Intersection of spheres encoded in a matrix: $A\in \mathrm{GL}(\mathbb{Z}[\pi_1 M])$



Theorem (s-cobordism, Barden-Mazur-Stallings, 1965):

{h-cobordisms from M } —Whitehead torsion / bijection→ Whitehead group $\mathrm{Wh}(\pi_1 M)$

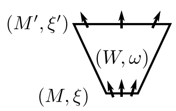


Example : $L(7,1)\times\mathrm{S}^2$ and $L(7,2)\times\mathrm{S}^2$ are not diffeomorphic (distinct Reidemeister torsion) but they are h-cobordant. Hence, $L(7,1)\times\mathrm{S}^2\times\mathbb{R}$ and $L(7,2)\times\mathrm{S}^2\times\mathbb{R}$ are diffeomorphic.

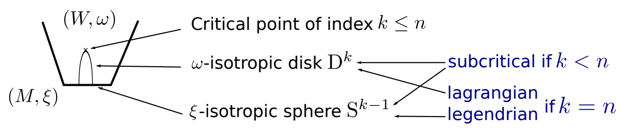
Conclusions:
1. **In high dimension, diffeomorphic products by $\mathbb{R}$ = h-cobordant.**
2. **In general, h-cobordant does not imply diffeomorphic.**
3. **After connect summing with sufficiently many $\mathrm{S}^2\times\mathrm{S}^{m-2}$, h-cobordant manifolds become diffeomorphic.**

Flexibility of symplectic structures

Symplectic cobordism (W,ω) of dimension $2n$ near the boundary = symplectization



Some have compatible handle decompositions: Weinstein cobordisms



Question: How to construct and deform Weinstein structures on cobordisms ?

spheres —deform→ isotropic spheres

- Subcritical spheres: Gromov's h-principle.
- Legendrian spheres: stabilization trick, Eliashberg (high dimensional).

Theorem (Eliashberg, 1990): (M,ξ) of dimension ≥ 5. W cobordism with handles of index $\leq \frac{1}{2}\dim W$ (+ homotopical condition), then W admits a Weinstein structure.

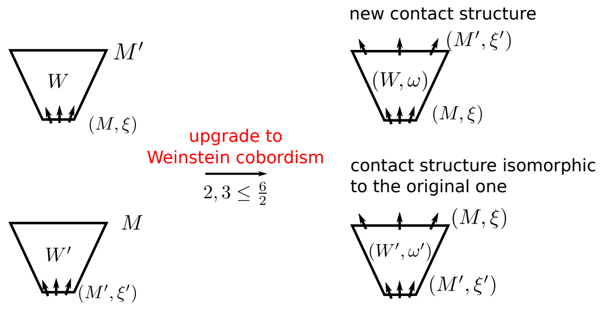
isotropic sphere ↓ smooth isotopy ↓ isotropic sphere —deform→ isotropic sphere ↓ isotropic isotopy ↓ isotropic sphere

- Subcritical spheres: Gromov's h-principle.
- Legendrian spheres: Restrict to loose ones, Murphy's h-principle.

⟶ define Flexible Weinstein cobordisms.

Theorem (Cieliebak-Eliashberg, 2012): (W,ω) flexible Weinstein cobordism of dimension ≥ 6. Any homotopy of the Morse function with critical points of index $\leq\frac{1}{2}\dim W$ can be realized by a Weinstein homotopy.

Main result



Theorem (C., 2013): (M,ξ) of dim ≥ 5, M' such that $M\times\mathbb{R}\simeq M'\times\mathbb{R}$ then there is a contact structure ξ' on M' sastisfying:
1) $(\mathrm{S}_\xi M,\omega_\xi)$ and $(\mathrm{S}_{\xi'}M',\omega_{\xi'})$ are symplectomorphic.
2) For $p\gg 1$, $M\#p\mathrm{S}^2\times\mathrm{S}^{m-2}$ and $M'\#p\mathrm{S}^2\times\mathrm{S}^{m-2}$ are contactomorphic.

Example: $M=L(7,1)\times\mathrm{S}^2\simeq\mathrm{ST}^*L(7,1)$, ξ = standard contact structure. M'= $L(7,2)\times\mathrm{S}^2$. By the theorem above, there is ξ' such that $(\mathrm{S}_\xi M,\omega_\xi)$ and $(\mathrm{S}_{\xi'}M',\omega_{\xi'})$ are symplectomorphic, though M and M' are not diffeomorphic !

Contact manifolds with symplectomorphic symplectizations need not be diffeomorphic!