Exploding trousers and computing the intractable

An introduction to scattering symplectic geometry

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Background.

My research is situated in an area of differential geometry called Poisson geometry. The simplest example of a Poisson manifold is called a symplectic manifold.

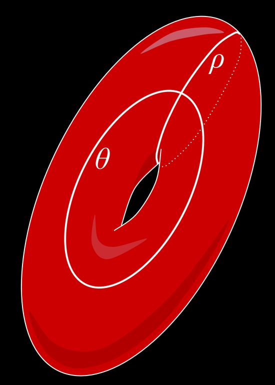
Symplectic geometry

Definition. A *symplectic manifold* is a manifold M with a closed, non-degenerate 2-form ω.

Example. Consider the torus $\mathbb{T}^2 \simeq \mathbb{S}^1 \times \mathbb{S}^1$ with coordinates θ and ρ. The 2-form

$$\omega = d\theta \wedge d\rho$$

is a symplectic structure.



Why study symplectic geometry?
Symplectic geometry has its origins in the Hamiltonian formulation of classical mechanics. In particular, the phase space of certain systems is a symplectic manifold.

Riemannian gradient versus symplectic gradient.
Given a Riemannian manifold (M, g), *the gradient of f* is the vector field

$$df = g(\,\cdot\,, \nabla f).$$

At each point, the gradient of f will show the direction in which the function changes most quickly. The symplectic gradient is the analog in symplectic geometry of the gradient in Riemannian geometry. Given a symplectic manifold (M, ω), *the symplectic gradient* $\nabla_\omega f$ is the vector field satisfying

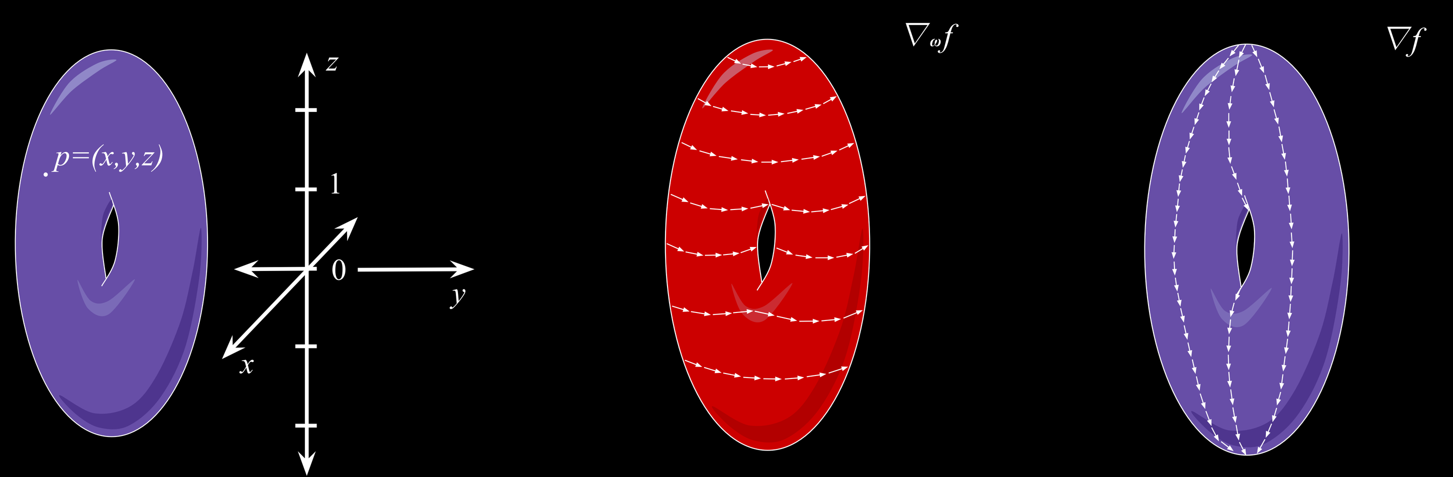
$$df = \omega(\,\cdot\,, \nabla_\omega f).$$

At each point, the symplectic gradient of f will show the direction in which the function changes least quickly. If you are interested in conserving quantities, symplectic geometry can be a powerful setting for studying a physical system.

Example. Consider the torus embedded into $\mathbb{R}^3$. Define a function

$$f : \mathbb{T}^2 \to \mathbb{R}$$

by $f(p) = z$. In other words, f is the height function.



embedding

symplectic gradient

gradient

Limitations. Unfortunately, many manifolds simply do not admit a symplectic form. For instance, no spheres $\mathbb{S}^{2n}$ for $n \geq 2$ and no odd dimensional manifolds can be symplectic. The next best type of structure is called a Poisson bi-vector.

Poisson geometry

There are many equivalent ways to define a Poisson structure.

Definition one. A *Poisson structure* on a manifold M is a symplectic (singular) foliation of M, i.e. a partition of M into symplectic manifolds (of possibly varying dimension) that fit together 'nicely'.

Example. Real three space $\mathbb{R}^3$ has many different Poisson structures.

Definition two. A *Poisson structure* on a manifold M is a bivector $\pi \in \mathcal{C}^\infty(M; \wedge^2 TM)$ satisfying the non-linear partial differential equation

$$[\pi, \pi] = 0.$$

(The bracket $[\cdot, \cdot]$ is a graded Lie bracket defined on multi-vector fields that extends the standard Lie bracket on vector fields.)

Returning to the symplectic case. By contracting in vector fields, a symplectic form ω provides an isomorphism of the tangent and cotangent bundles. The inverse to this map defines a Poisson bi-vector.

$$TM \underset{\pi^\sharp}{\overset{\omega^\flat}{\rightleftarrows}} T^*M$$

Consequently, symplectic structures are equivalent to non-degenerate Poisson structures. In general, π can be very degenerate. Note $[\pi, \pi] = 0$ is equivalent to $d\omega = 0$.

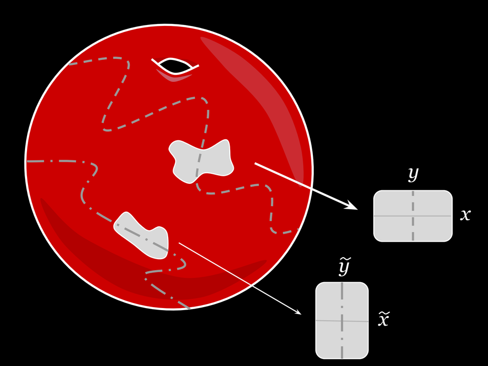
Symplectic versus Poisson. Symplectic structures are quite well understood. For instance, ALL symplectic forms on an n-dimensional manifold locally look the same. Poisson structures on the other hand can be much more gnarly to work with. In general we cannot even state a local normal form for a Poisson bi-vector π.

Research program.

I study 'minimally degenerate' Poisson structures, bi-vectors that have some degeneracy but that are very close to being symplectic.

Set up. Take (M, π) Poisson and a hypersurface $Z \subset M$. A hypersurface is a subspace of M that locally looks like the set $\{x_1 = 0\}$ in $\mathbb{R}^n$ for standard coordinate x_1. Demand that

- π is symplectic on $M \setminus Z$
- π is degenerate in a particular way at Z.



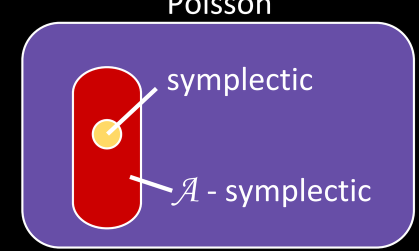
Goal. Think of π as non-degenerate on a new vector bundle $\mathcal{A}$.

$$\mathcal{A} \underset{\pi^\sharp}{\overset{\omega^\flat}{\rightleftarrows}} \mathcal{A}^*$$

Examples.

bundle	vector fields	co-vectors
${}^bTM$	$x\partial_x, \partial_{y_1}, \partial_{x_2}, \partial_{y_2}$	$\frac{dx}{x}, dy_1, dx_2, dy_2$
${}^0TM$	$x\partial_x, x\partial_{y_1}, x\partial_{x_2}, x\partial_{y_2}$	$\frac{dx}{x}, \frac{dy_1}{x}, \frac{dx_2}{x}, \frac{dy_2}{x}$
${}^{sc}TM$	$x^2\partial_x, x\partial_{y_1}, x\partial_{x_2}, x\partial_{y_2}$	$\frac{dx}{x^2}, \frac{dy_1}{x}, \frac{dx_2}{x}, \frac{dy_2}{x}$

Why do this? Viewing π as non-degenerate on $\mathcal{A}$ allows us to employ symplectic tools.



Note. A bi-vector that is non-degenerate on one of these $\mathcal{A}$'s corresponds to a 'singular' symplectic-type structure.

Scattering symplectic geometry

Definition. Let M be a manifold with a hypersurface $Z = \{x = 0\}$. A *scattering symplectic structure* on M is a closed non-degenerate section of the second exterior power of ${}^{sc}T^*M$. Given a point $p \in Z$, any scattering symplectic structure ω locally has the form

$$\omega = \frac{dx}{x^3} \wedge \alpha - \frac{d\alpha}{2x^2}$$

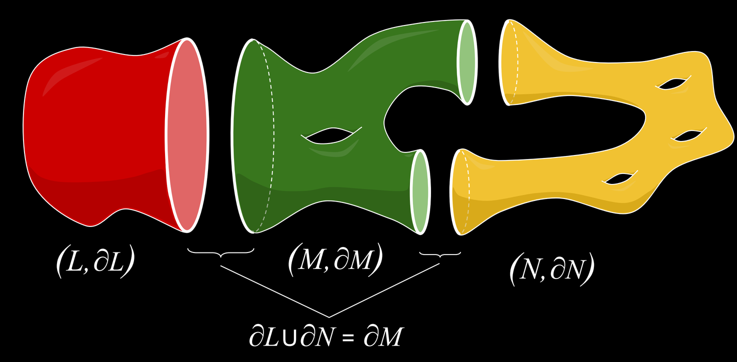
where α is a contact form on Z.

Results and techniques.

While a scattering symplectic form ω is singular, this structure corresponds to an actual smooth Poisson structure. Formally allowing this singularity and doing symplectic geometry in this generalized setting gives us a LOT of traction.

Cobordism

Cobordism is an equivalence relation between compact manifolds of the same dimension: two n-dimensional manifolds are cobordant if their disjoint union is the boundary of a compact manifold of dimension $n + 1$.



There are many different flavors of cobordism. Symplectic geometers study cobordisms between contact manifolds (Z, α) where the 'filling' is symplectic and satisfies a nice relationship with α. We showed:

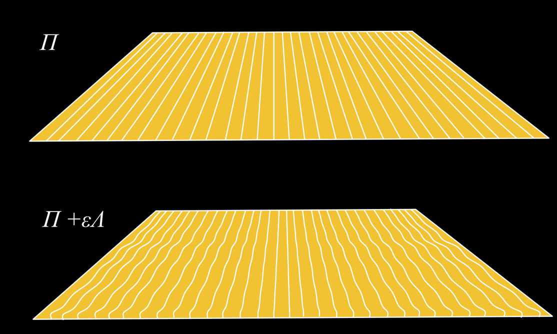
Theorem [L]. Given (M_1, ω_1), (M_2, ω_2) strong convex symplectic fillings of (Z, α), then $M_1 \cup_Z M_2$ admits a sc-symplectic structure ω.

Application. $(\mathbb{S}^{2n}, \mathbb{S}^{2n-1})$, $(\mathbb{T}^2 \times \mathbb{S}^2, \mathbb{T}^3)$, and $(\mathbb{S}^3 \times \mathbb{S}^1, \mathbb{S}^2 \times \mathbb{S}^1)$.

Poisson cohomology

Poisson cohomology is an invariant of a Poisson manifold (M, π). Each cohomology group $H^n_\pi(M)$ that we associate to (M, π) has an interpretation analogous to the way we say "the n^{th} singular cohomology group counts the number of n-dimensional holes in a manifold". We interpret $H^2_\pi(M)$ as the quotient of the space of all possible infinitesimal deformations of π by the space of trivial deformations.

In other words, $H^2_\pi(M)$ is supposed to count the number of Poisson structures nearby that are actually different from π. Accordingly, this invariant has the potential to tell us a lot about our Poisson structure, particularly about local normal forms.



Downside. Unfortunately, there are very few known explicit computations. When π is non-degenerate, i.e. symplectic, the Poisson cohomology is isomorphic to the de-Rham cohomology of M. This isomorphism is given by a map induced from ω.

Our method. Inspired by this isomorphism, we use scattering symplectic ω to establish an isomorphism with a de-Rham like complex. In essence, we take a really hard problem and use the scattering-symplectic structure to turn it into a much more easy computation. We have successfully used this approach to compute the Poisson cohomology of many types of 'minimally degenerate' Poisson structures.

Preprints. *Symplectic, Poisson, and contact geometry on scattering manifolds.* arXiv:1603.02994

Poisson cohomology of a class of log symplectic manifolds. arXiv:1605.03854

Acknowledgement. *Travel support was provided through grants from the National Science Foundation (NSF) including the ADVANCE grant Career Advancement for Women Through Research-Focused Networks, and Graduate Student Participation in National Workshops to Encourage Women's Engagement in Mathematics Research.*